COMPUTING ARITHMETIC SERIES BY EXPRESSING THE SUMMANDS AS A SUM OF ONES

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Abstract. This note considers the computation of arithmetic series. The aim is to derive an alternative approach to finding the formula by expanding the sum into Ones and finding geometric patterns. The general validity of the formula can then be proven by induction.

1. Arithmetic series

In the following we will explore the approach of expanding the summands of an arithmetic series as a sum of Ones[1] in order to derive the formula for the induction start. Since for the cases considered in the following, the formulas are already known, we skip the induction step and compare the formula derived from the geometric patterns to the known formulas to verify the approach.

2. Expansion of arithmetic series

2.1. **Gaußian summation formula.** Consider the well known Gaußian summation formula (1).

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \ . \tag{1}$$

2.2. **Interpretation as trapezoid.** We consider the simple Gaußian summation formula.

Let us optically analyze the pattern of the sum expanded into a sum of Ones, see Figure 1. We observe, that the Ones are arranged in a pattern forming an orthogonal "triangle" with side length of n.

$$\sum_{k=1}^{n} k = \quad 1 \ + \ 2 \ + \ 3 \ + \cdots + \ n \quad = \frac{n(n+1)}{2}$$

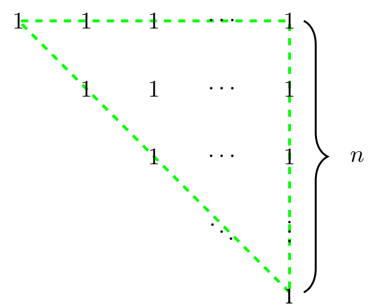

Figure 1. Geometric pattern of the Gaußian sum expanded into a sum of Ones.

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Assume that each One represents a distance of length 1. Then this sum can be geometrically repesented as shown in Figure 2. The area of the trapezoid given by the points A, B, C and D in Figure 2 coincides with the value of the Gaußian sum.

$$\sum_{k=1}^{n} k \cong F_{\text{trapezoid}} = \frac{\overline{AB} + \overline{CD}}{2} h = \frac{1 + \text{n}}{2} \text{n} ,$$

which coincides with the result from the direct computation formula of the Gaußian sum (1)

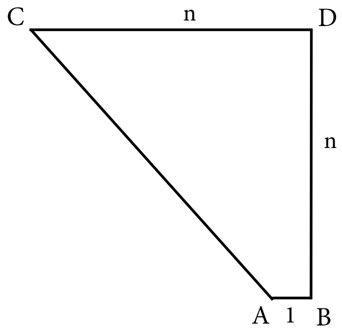

FIGURE 2. Gaußian sum coincides with the area of a trapezoid.

2.3. **Using geometric patterns to discover formulas for more general arithmetic series.** Consider next the sum of every second odd number,

$$\sum_{k=1}^{n} (4k - 3) = 1 + 5 + 9 + \ldots + (4n - 3) \ .$$

By using simple arithmetic rules, one directly obtains the following formula,

$$\sum_{k=1}^{n} (4k - 3) = 4 \sum_{k=1}^{n} k - 3n = 2n^2 - n \ .$$

However, by following our new geometric approach we expand this sum in sums of Ones as shown in Figure 3. There we do a little rearranging of the summands and then recognize the patterns from the original Gaußian sum as well as other geometric objects. Now, we add the three formulas that we obtain

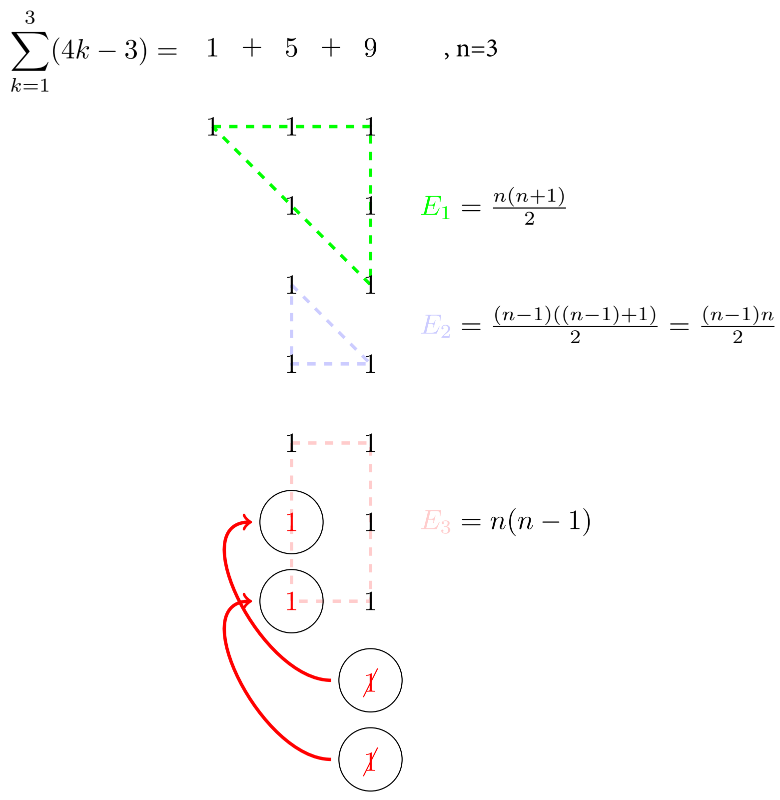

FIGURE 3. Geometric pattern of the sum over every second odd number expanded into a sum of Ones.

from the recognized patterns, which yields

$$\sum_{k=1}^{n}(4k-3) = E_1(n) + E_2(n) + E_3(n)$$
$$= \frac{n(n+1)}{2} + \frac{n(n-1)}{2} + n(n-1) = 2n^2 - n \ .$$

Observe that the geometric interpretation leads to the correct formula for the sum.

Another example for this approach is the following sum,

$$\sum_{k=1}^{n}(3k-2) = 1 + 4 + 7 + \ldots + (3n-2) = 3\sum_{k=1}^{n} k - 2n = \frac{3n^2-n}{2} \ .$$

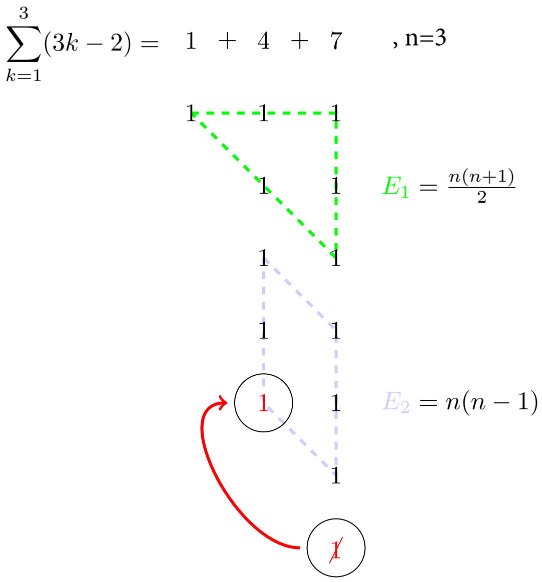

FIGURE 4. Geometric pattern of the sum over every third number expanded into a sum of Ones.

Similarly to the above example we expand this sum in sums of Ones as shown in Figure 4. There we do a little rearranging of the summands and then recognize the patterns from the original Gaußian sum as well as other geometric objects. Now, we add the three formulas that we obtain from the recognized patterns, which yields

$$\sum_{k=1}^{n}(3k-2) = E_1(n) + E_2(n) = \frac{n(n+1)}{2} + n(n-1) = \frac{3n^2-n}{2} \ .$$

Once again, the recognition of the geometric patterns leads to the correct formula for the sum.

3. SECOND ORDER ARITHMETIC SERIES

Consider the second order arithmetic series

$$\sum_{k=1}^{n} k^2 = 1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \ , \tag{2}$$

which is a special case of the Faulhaber's Formula, see e.g. [2,4].

For the example of $n = 3$, we again expand this sum into a sum of Ones, see Figure 5. The pattern indicated in the expansion of the sum into Ones motivates the following definition. Consider a rectangular triangle ABC with the catheti $\overline{AB} = 3$ and $\overline{AC} = 4$. Using this triangle as base area, we further define a pyramid by placing another point D directly above A, such that the height $\overline{AD} = 7$ is the sum of the

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$
$$= 1 \ + \ 4 \ + \ 9$$

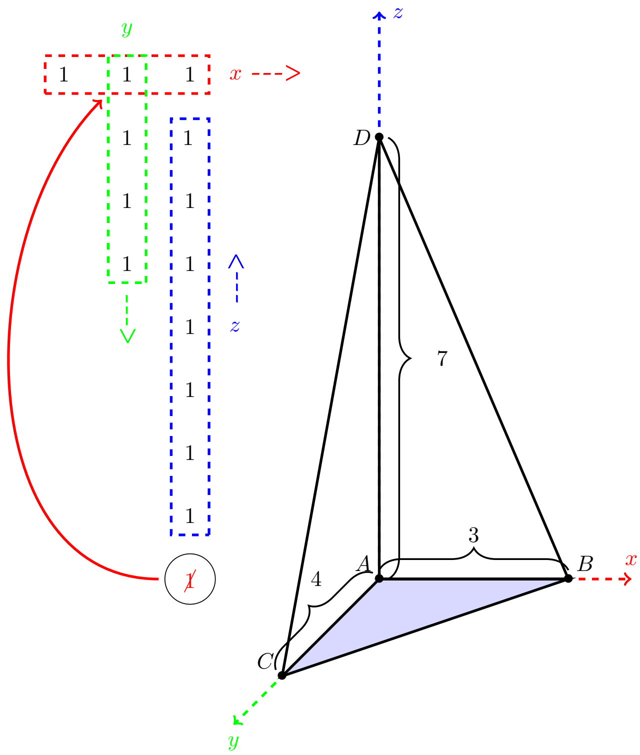

FIGURE 5. Pyramid with the volume given by the second order arithmetic sum.

two catheti of the base triangle, see Figure 5. Now, we compute the volume of this pyramid. Using basic geometric formulae, we obtain

$$F_{\text{base}} = \frac{1}{2}\overline{AB} \cdot \overline{AC} = \frac{1}{2} \cdot 3 \cdot 4 = 6 \ ,$$
$$V_{\text{pyramid}} = \frac{1}{3}h \cdot F_{\text{base}} = \frac{1}{3}\overline{AD} \cdot \frac{1}{2}\overline{AB} \cdot \overline{AC} = \frac{1}{3} \cdot 7 \cdot \frac{1}{2} \cdot 3 \cdot 4 = 14 \ .$$

Observe that the volume of the pyramid coincides with the second order arithmetic sum up to $n = 3$,

$$\sum_{k=1}^{3} k^2 = 1 + 4 + 9 = 14 \ .$$

We now apply this procedure more generally: Consider a rectangular triangle ABC with the catheti $\overline{AB} = n$ and $\overline{AC} = n + 1$. Using this triangle as base area, we further define a pyramid by placing another point D directly above A, such that the height $\overline{AD} = n + (n + 1)$ is the sum of the two catheti of the base triangle, see Figure 6. Again, we compute the volume of this pyramid,

$$F_{\text{base}} = \frac{1}{2}\overline{AB} \cdot \overline{AC} = \frac{1}{2}n(n + 1) \ ,$$
$$V_{\text{pyramid}} = \frac{1}{3}h \cdot F_{\text{base}} = \frac{1}{3}\overline{AD} \cdot \frac{1}{2}\overline{AB} \cdot \overline{AC} = \frac{1}{3}(n + (n + 1)) \cdot \frac{1}{2}n(n + 1) \ .$$

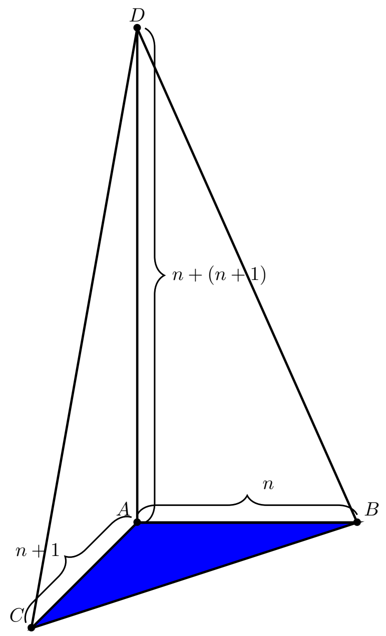

FIGURE 6. Pyramid with the volume given by the second order arithmetic sum.

This formula also coincides with the general formula for the sum of squares,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}(2n+1) \cdot \frac{1}{2}n(n+1) = V_{\text{pyramid}} \ .$$

This is shown in Figure 6.

Note, while the definition of $\overline{AB} = n$ is rather obvious, that considering the case $n = 3$ as motivation one could also choose the definitions $\overline{AC} = (n-1)^2$ and $\overline{AD} = n^2 - 2$. This however only yields the correct value for the case $n = 3$. Moreover, due to the kind of sum under consideration and the aim to define a three-dimensional object, it makes sense to choose the dependency of $\overline{AC}$ and $\overline{AD}$ linear in n as we proposed in the example above and which does yield the correct result.

4. More general arithmetic sequences

Consider a more general definition of an arithmetic sequence defined by the recursive formula

$$a_1 = a \ , \quad a_n = a_{n-1} + d(n-1) \ , \quad n = 2, 3, 4, \ldots$$

for an arbitrary function d. For this definition, there is also a representation of the n-th element of the sequence as a direct formula.

$$\begin{aligned} a_n &= a_{n-1} + d(n-1) = a_{n-2} + d(n-2) + d(n-1) = \ldots \\ &= a_1 + d(1) + d(2) + \ldots + d(n-1) = a + \sum_{l=1}^{n-1} d(l) \end{aligned}$$

Note that the sum over $d(k)$ is a coarse approximation to the integral over d,

$$\sum_{l=1}^{n-1} d(l) \approx \int_0^{n-1} d(x) \ dx \ .$$

Now consider again the sum over the general arithmetic sequence:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(a + \sum_{l=1}^{k-1} d(l) \right) = na + \sum_{k=1}^{n} \sum_{l=1}^{k-1} d(l) \ .$$

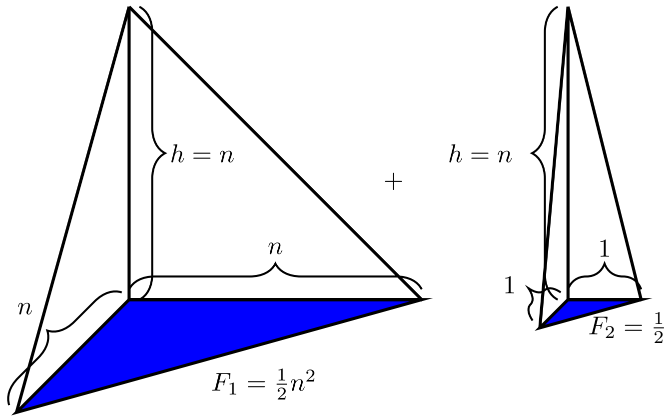

FIGURE 7. The volume of the two pyramids together equals the value of the general arithmetic sum with $a_1 = 1$ and $d(n) = n$.

Consider $a = 1$ and $d(n) = n$, then the following formula holds,

$$\begin{aligned} \sum_{k=1}^{n} a_k &= n + \sum_{k=1}^{n}\sum_{l=1}^{k-1} l = n + \sum_{k=1}^{n} \frac{k(k-1)}{2} = n + \frac{1}{2}\sum_{k=1}^{n}(k^2 - k) \\ &= n + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right) \\ &= n + \frac{1}{2}\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - 1\right) \\ &= \frac{n^3+5n}{6} = \frac{1}{3}\left(\frac{1}{2}\cdot n \cdot n\right) n + \frac{1}{3}\left(\frac{1}{2}\cdot 1 \cdot 1\right) n \ . \end{aligned}$$

This formula can be interpreted as the sum of the volumes of two triangular pyramids, each with height n, the first one with base area $F_1 = \frac{1}{2}\cdot n \cdot n$ and the second one with base area $F_2 = \frac{1}{2}\cdot 1 \cdot 1$, see Figure 7.

5. Conclusion

For first order general arithmetic sums we outlined a method to obtain formulas for the direct computation of the sums by recognizing patterns in the expansion of the sums into Ones that resemble simpler geometric objects such as the Gaußian sum expanded into a sum of Ones or rectangular arrays of Ones. Using these observations as well as the formulas for the simpler geometric objects we derived the formula for the more general arithmetic sum. Further, we gave a geometric interpretation of second order arithmetic sums as pyramids and the sum over summands grow with a linear increment as a sum of two pyramids.

Acknowledgement

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